**Lecture 7**

**DEFINITION OF A SEQUENCE**

In everyday language, the term “sequence” means a succession of things in a definite order-chronological order, size order, or logical order, for example. In mathematics, the term “sequence” is commonly used to denote a succession of numbers whose order is determined by a rule or a function. We will develop some of the basic ideas concerning sequences of numbers.

Stated informally, an ***infinite sequence***, or more simply a ***sequence***, is an unending succession of numbers, called ***terms***. It is understood that the terms have a definite order; that is, there is a first term *a*1, a second term *a*2, a third term *a*3, a fourth term *a*4, and so forth.

Such a sequence would typically be written as

*a*1*, a*2*, a*3*, a*4*, . . .*

where the dots are used to indicate that the sequence continues indefinitely. Some specific examples are

1*,* 2*,* 3*,* 4*, . . . ,* 1*,* *,* *,* *, . . . ,*

2*,* 4*,* 6*,* 8*, . . . ,* 1*,*−1*,* 1*,*−1*, . . .*

Each of these sequences has a definite pattern that makes it easy to generate additional terms if we assume that those terms follow the same pattern as the displayed terms. However, such patterns can be deceiving, so it is better to have a rule or formula for generating the terms. One way of doing this is to look for a function that relates each term in the sequence to its term number. For example, in the sequence

2*,* 4*,* 6*,* 8*, . . .*

each term is twice the term number; that is, the *n-*th term in the sequence is given by the formula 2*n*. We denote this by writing the sequence as

2*,* 4*,* 6*,* 8*, . . . ,* 2*n, . . .*

We call the function *f(n)* = 2*n* the *general term* of this sequence. Now, if we want to know a specific term in the sequence, we need only substitute its term number in the formula for the general term.

For example, the 37-th term in the sequence is 2・37 = 74.

**INFINITE SERIES**

The purpose of this section is to discuss sums that contain infinitely many terms. The most familiar examples of such sums occur in the decimal representations of real numbers.

For example, when we write  in the decimal form  = 0*.*3333 *. . . ,* we mean

= 0*.*3 + 0*.*03 + 0*.*003 + 0*.*0003+…

which suggests that the decimal representation of  can be viewed as a sum of infinitely many real numbers.

**SUMS OF INFINITE SERIES**

Our first objective is to define what is meant by the “sum” of infinitely many real numbers.

We begin with some terminology.

**Definition.** An ***infinite series*** is an expression that can be written in the form

 (1)

The numbers *а1*, *а2*, …, *аn*, … are called the ***terms*** of the series.

Let *Sn* denote the sum of the initial terms of the series, up to and including the term with index *n*. Thus,





… (2)



The number *Sn* is called the ***n-th partial sum*** of the series and the sequence

*S1*, *S2*, ..., *Sn*, ...  (3)

is called the ***sequence of partial sums***.

As *n* increases, the partial sum  includes more and more terms of the series. Thus, if *Sn* tends toward a limit as , it is reasonable to view this limit as the sum of *all* the terms in the series. This suggests the following definition.

**Definition.** Let  be the sequence of partial sums of the series (1).

If the sequence  converges to a limit *S*, , then the series is said to ***converge*** to *S*, and *S* is called the ***sum*** of the series. We denote this by writing



If the sequence of partial sums diverges, then the series is said to ***diverge***. A divergent

series has no sum.

**Example 1.** Determine whether the series:



converges or diverges. If it converges, find the sum.

*Solution*. We turn directly to Definition. The partial sums are



For any sum is true following things:

  ..., 

This can be accomplished by using the method of partial fractions to obtain (verify)

We rewrite *Sn*:



.

Then

.

The sequence  converges, then the series is convergeto *S=1*.

In many important series, each term is obtained by multiplying the preceding term by some fixed constant. Thus, if the initial term of the series is *a* and each term is obtained by multiplying the preceding term by *q*, then the series has the form

 . (5)

Such series are called ***geometric series***, and the number *q* is called the ***ratio*** for the series.

Here are some particular cases:

a) If  then ,  and . So {*Sn*} converges and .

b) If  then



and  және . So {*Sn*} converges and .

If *q* = 1, then the series is



so the *n-*th partial sum is and



This proves divergence.

If *q* = −1, the series is



so the sequence of partial sums is

*a,* 0*, a,* 0*, a,* 0*, . . .*

which diverges.

Now let us consider the case where,  , , . The *n-*th partial sum of the series is

 (6)

Multiplying both sides of (6) by *q* yields

 (7)

and subtracting (7) from (6) gives



or

 (8)

This can be rewritten as

 (9)

1) If |*q*| *<* 1, then , so {*Sn*} converges. From (9)



2) If |*q*| *>* 1, then , so {*Sn*} diverges

**Theorem.** *A geometric series*



*converges if* |*q*| *<* 1 *and diverges if* |*q*| ≥ 1*. If the series converges, then the sum is*



***Example 2***. Determine whether the series

0*.*3 + 0*.*03 + 0*.*003 + 0*.*0003+…

converges or diverges. If it converges, find the sum.

**Algebraic properties of infinite series**

For brevity, the proof of the following result is omitted.  
1. *Convergence or divergence is unaffected by deleting a finite number of terms from*

*a series; in particular, for any positive integer m, the series* and  *both converge or both diverge.*

2. *If c is a nonzero constant, then the series* *and*  *both converge or both*

*diverge. In the case of convergence, the sums are related by*

 .

3. *If*  *and*  *are convergent series, then*  *are convergent series and the sums of these series are related by* .

4*. If at least one of the series  and  is diverge, then are diverges series.*

5. *If*  *and*  *are diverge series, then*  *are can convergent or diverge.*

**2. CONVERGENCE TESTS**

In this section we will develop various tests that can be used to determine whether a given series converges or diverges.

**Theorem (*The necessary test of convergencet*)**. *If the series* *******converges, then*

.

Proof. Let us assume that the series converges. The general term *an*can be written as

. (1)

If *S* denotes the sum of the series, then , also have. Thus, from (1)

.

**Theorem (*The Divergence Test*)**. *If* *, then the series* ******  *diverges.*

**Example**. The series

.

### *Solution*. The general term is .

We check necessary condition for convergence:

,

The series is diverge.

**Example***.* One of the most important of all diverging series is the ***harmonic series***,

 (7)

which arises in connection with the overtones produced by a vibrating musical string. It is not immediately evident that this series diverges. However, the divergence will become apparent when we examine the partial sums in detail. Because the terms in the series are all positive, the partial sums

We check necessary condition for convergence:

.

But this series is divergent. We shall show that. If the series converges, then we would have had the sum S and can write



But,



As well  there can not be . This means that the series diverges.

**The sufficient conditions for the convergence**

10. **The comparison test.** We will begin with a test that is useful in its own right and is also the building block for other important convergence tests. The underlying idea of this test is to use the known convergence or divergence of a series to deduce the convergence or divergence of another series.

**Theorem (*The Comparison Test*)** *Let*  *and* *be series with nonnegative*

*terms and suppose that* for any 

(*a*) *If the “bigger series”*  *converges, then the “smaller series”*  *also converges.*

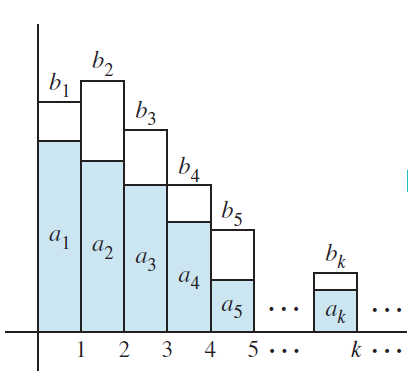
(*b*) *If the “smaller series”* *diverges, then the “bigger series”*  *also diverges.*

The proof is easy to visualize why the theorem is true by interpreting the terms in the series as areas of rectangles (Figure 1.)

The comparison test states that if the total area  is finite, then the total area

must also be finite; and if the total area is infinite, then the total area

 must also be infinite.



**Figure 1.**

For each rectangle, *an* denotes the area of the blue portion and *bn* denotes the combined area of the white and blue portions.

***Using the comparison test***

There are two steps required for using the comparison test to determine whether a series  with positive terms converges:

**Step 1.** Guess at whether the series converges or diverges.

**Step 2.** Find a series that proves the guess to be correct. That is, if we guess that  diverges, we must find a divergent series whose terms are “smaller” than the

corresponding terms of , and if we guess that converges, we must find a convergent series whose terms are “bigger” than the corresponding terms of .

In most cases, the series  being considered will have its general term *an* expressed as a fraction. To help with the guessing process in the first step, we have formulated two principles that are based on the form of the denominator for *an*. These principles sometimes suggest whether a series is likely to converge or diverge. We have called these “informal principles” because they are not intended as formal theorems. In fact, we will not guarantee that they always work. However, they work often enough to be useful.

*Informal principle 1*. Constant terms in the denominator of *an* can usually be deleted without affecting the convergence or divergence of the series.

*Informal principle 2*. If a polynomial in *n* appears as a factor in the numerator or denominator of *an*, all but the leading term in the polynomial can usually be discarded without affecting the convergence or divergence of the series.

***Example****.*  Determine whether the series converges or diverges



*Solution.* We will guess that the given series converges and try to prove this by finding a convergent series that is “bigger” than the given series. We take the geometric series with the ratio*q=1/3*.

 (\*)

So, series (\*) does the trick since

, , ...,  , ...

Thus, we have proved that the given series converges.

***Example****.*  Determine whether the series converges or diverges



*Solution.* We will guess that the given series diverges and try to prove this by finding a divergent series that is “smaller” than the given series. We take the harmonic series

 (\*\*)

However, series (\*\*) does the trick since:

, , ...,  , ...

Because, , , ..., . Thus, we have proved that the given series diverges.

**The limit comparison test**

In the last two examples, we have made the guess about convergence or divergence as well as the series needed to apply the comparison test. Unfortunately, it is not always so straightforward to find the series required for comparison, so we will now consider an alternative to the comparison test that is usually easier to apply.

**Theorem (*The Limit Comparison Test*)** *Let*  *and* *be series with positive*

*terms and suppose that* 

*If ρ is finite and ρ >* 0*, then the series both converge or both diverge.*

2°. THE RATIO TEST or D'ALEMBERT TEST

The comparison test and the limit comparison test hinge on first making a guess about convergence and then finding an appropriate series for comparison, both of which can be difficult tasks in cases where Informal principle*s* 1 and 2 cannot be applied. In such cases the next test can often be used, since it works exclusively with the terms of the given series - it requires neither an initial guess about convergence nor the discovery of a series for comparison.

**Theorem (*The Ratio Test*)** *Let*  *be a series with positive terms and suppose that*



(*a*) *If l <* 1*, the series converges.*

(*b*) *If l >* 1 *or ρ* = *, the series diverges.*

(*c*) *If l* = 1*, the series may converge or diverge, so that another test must be tried.*

**7-мысал***.*  Test the given series for convergence



Шешуі. *n*-th and -th terms are , . The series converges, since

.

And .

3°. **Theorem (*The Cauchy Integral Test*).**  *Let*  *be a series with positive terms. If f(x) is a function that is decreasing and continuous on an interval*  *and such that* , , , ... *for all n* ≥ *1, then*  *and*  *both converge or both diverge.*

**Example***.*  Show that the integral test applies, and use the integral test to determine

whether the ***hyperharmonic series*** converge or diverge

 (8)

*Solution*. We will use the integral test. Here  ,  And  for all  Assume first that *p >* 1. меншіксіз интеграл жинақтылығымен бірдей болады.

If , then .

If  then



  *or* hyperharmonic series *converges if*  *and diverges if* *.*

**The Cauchy Root Test**

In cases where it is difficult or inconvenient to find the limit required for the ratio test, the next test is sometimes useful. Since its proof is similar to the proof of the ratio test, we will omit it.

**Theorem (*The Cauchy Root Test*).**  *Let*  *be a series with positive terms and suppose that*



(*a*) *If l <* 1*, the series converges.*

(*b*) *If l >* 1 *or ρ* = *, the series diverges.*

(*c*) *If l* = 1*, the series may converge or diverge, so that another test must be tried.*

**Example***.*  Test the given series for convergence



***Solution.*** The series diverges, since

